

Symbolic dynamics I Finite dispersive billiards.

Kai T. Hansen
*Niels Bohr Institute*¹
Blegdamsvej 17, DK-2100 Copenhagen Ø
e-mail: khansen@nbivax.nbi.dk

ABSTRACT

Orbits in different dispersive billiard systems, e.g. the 3 disk system, are mapped into a topological well ordered symbol plane and it is showed that forbidden and allowed orbits are separated by a monotone pruning front. The pruning front can be approximated by a sequence of finite symbolic dynamics grammars.

¹ ‡ Permanent address: Phys Dep., University of Oslo, Box 1048, Blindern, N-0316 Oslo

1 Introduction

The aim of this article is to find an effective description of all allowed orbits in a classical billiard system. The systems considered are a point particle on a plane, moving freely and reflecting elastically off dispersing walls. In most of our examples the walls are the borders to circular shaped reflecting disks. Such systems have recently been in focus of much theoretical interest, because of their utility in studies of the quantum mechanics of classical chaotic systems [1, 2, 3]. Semiclassical calculations of the quantal spectrum of classically chaotic systems require summations over all periodic orbits of the classical system^[4], and it is essential to have an effective description of the allowed classical orbits.

The solution we propose here is to apply techniques developed in the study of one dimensional unimodal maps, and of two dimensional Hénon-type maps. In the unimodal maps the ordering of allowed orbits along the parameter axis is given by the MSS theory [5], i.e. the kneading sequence of the critical point^[6]. From the kneading sequence one constructs a binary number γ_c and the interval $(\gamma_c, 1)$ is the primary pruned region of the unimodal map. An orbit is forbidden if the orbit, or any of its images or preimages yields a binary number γ in the pruned region $(\gamma_c, 1)$. A convenient visualization of the allowed orbits is afforded by a binary tree on which every allowed orbit corresponds to a path down the tree. The forbidden orbits correspond to the forbidden branches, i.e. the branches that have to be *pruned* [7] from the tree of allowed orbits.

A unimodal map such as the logistic map has a parameter space window for each stable periodic orbit. Within such window the kneading sequence is periodic, one can find a finite number of forbidden symbol sub-sequences, and construct a new unpruned alphabet^[8]. One can also describe allowed sequences as walks on a finite Markov graph constructed from the pruned binary tree^[9]. Chaotic dynamics does not in general have a finite Markov partition, but we believe^[10] that the statistical properties can be approximated from calculations on repellers corresponding to converging sequences of nearby windows with stable orbits.

In ref.^[7] Cvitanović, Gunaratne and Procaccia have extended these methods to two dimensional maps like the Hénon map^[11] and showed that numerically one can obtain a connected primary pruned region in a two dimensional space (γ, δ) constructed from the properly ordered symbolic future and the past of the orbits; the border of the area is the pruning front. The conjecture is that this curve is monotone, and that there are no other forbidden orbits than those that pass through the primary pruning region. One can approximate the grammar of forbidden orbits by forbidding finite sequences that correspond to rectangles in the symbol plane. Each such approximation gives a finite number of finite length forbidden symbol sequences. From these sequences one can make successively refined Markov par-

titions and obtain well converging estimates of averages such as the topological entropy^[12, 13, 14]. The pruning front is the symbol plane representation of the primary homoclinic tangencies introduced by Grassberger and Kantz^[15] in order to partition the Hénon map non-wandering set into a “left” and a “right” side.

1.1 Symbols in billiard systems

In billiard systems the choice of symbols is much easier; a symbol for bounce off each concave wall, e.g. a disk or a branch of the hyperbola, is natural and unambiguous^[16, 17, 18, 19]. Billiards considered here are hyperbolic systems with only unstable orbits; the stable and unstable manifolds of the orbits are never tangential to each other. An orbit in the billiard is unstable also at the point where it is pruned, in contrast to the Hénon map and smooth Hamiltonian potentials, where pruning always involves inverse bifurcations and stable orbits^[20, 21].

Construction of billiard symbolic dynamics proceeds in 2 steps: 1. define a symbol plane that is topologically *well ordered*; 2. determine the *boundary orbits* between the legal and the illegal orbits. The well ordered symbol plane is defined by introducing a new symbol alphabet and associating with every orbit a pair of coordinates constructed from the alphabet. A similar well ordered symbolic alphabet has been introduced for a different scattering problem by Troll^[22]. The second issue is simple. We show that for the billiards one starts a tangential bounce from every point on the wall and these are the boundary orbits.

The symbol plane we construct is complete for a system with a finite number of hard disks situated far from each other in the configuration space. A physically motivated example is a particle bouncing between 4 hyperbolas^[3, 24]. This billiard do not have a obvious limit of a complete Cantor set but we use the same symbol plane as the 4-disks billiard and orbits are pruned the same way.

2 Covering alphabet for dispersing billiards

For simplicity we choose to describe the covering alphabets for dispersing billiards by using billiards with a finite number of identical circularly shaped walls. In general the walls do not have to be circular, but they do have to be dispersing. When they are identical, the configuration space symmetry enables us to reduce the number of symbols and to simplify the problem.

If we take a billiard system with *sufficiently separated* disks in the plane there exist an orbit which visits the disks in any desired order.

Let the disks be enumerated by $s \in \{1, 2, \dots, N\}$ and denote the bounce of the point particle on disk s at the discrete time t by symbol s_t . Dispersing (concave)

walls can not have two succeeding bounces off the same wall, i.e. the subsequence $_{ss}$ is forbidden, but all other sequences are allowed for sufficiently separated disks.

Given the infinite symbolic past of the particle bouncing on disk s_0 at time $t = 0$, that is $\dots s_{-3}s_{-2}s_{-1}$, and the infinite symbolic future $s_1s_2s_3\dots$, the position of the bounce x and the outgoing angle ϕ measured from the normal vector of the wall are uniquely determined. The phase space coordinates we use are the position and the angle (x, ϕ) of the bounce off disk s_0 .

If the N-disk system is open, then almost all points (x, ϕ) correspond to a particle that has entered the system from outside at some earlier time and is going to escape from the system at some later time. The union of all points contained within the billiard for all times is the non-wandering set Λ . For a sufficiently separated disk billiard, Λ is a Cantor set in the phase space similar to the non-wandering set for the Smale horseshoe [23]. Each point in this Cantor set is described by a unique symbolic past and future. However, if we use the disk symbols s_t to describe a point in the Cantor set, there is no obvious ordering of the symbols from “small” symbols to “large” symbols that would reflect an increase of the position or angle coordinate in the phase space. A symbolic description with a natural ordering of this kind we call well ordered symbols and we show how these can be constructed by a few examples. Other configurations of disks can be worked out in a similar way.

2.1 3 disks symbolic dynamics

Let the billiard system be 3 equal disks with radius 1 and the distance r between the centers, enumerated anticlockwise as in figure 1. Without loss of generality we assume that the particle bounces off disk number 1 at time $t = 0$ ($s_0 = 1$). The position x is the angle describing the bounce on disc 1 and ϕ is the angle from the normal vector. Define Λ_t^+ as the union of points in phase space that at least give t bounces in the billiard after bouncing off disc 1. Thus Λ_1^+ is all (x, ϕ) of disc 1 corresponding to a particle that bounces either off disk 2 or off disk 3 at time $t = 1$. Λ_1^+ consists of two diagonal strips in phase space, with ϕ decreasing when x is increasing. The lower strip in the phase space is the union of all points in phase space corresponding to a particle bouncing off disk 2 at time $t = 1$ and the symbol sequence for this strip is $s_0s_1 = 12$. The upper strip is the union of all points corresponding to a particle that bounces off disk 3 at time $t = 1$, and this strip has symbol sequence $s_0s_1 = 13$. The order of the two strips with increasing x or ϕ is $s_0s_1 = \{12, 13\}$. Λ_2^+ is four diagonal strips, two inside each of the two strips of Λ_1^+ . The symbol description of the four strips ordered after increasing values of x or ϕ is $s_0s_1s_2 = \{121, 123, 132, 131\}$. Table 1 shows the symbol sequences for 4 bounces ordered after increasing values of x and ϕ . The limit of Λ_t^+ when $t \rightarrow \infty$ is a Cantor set of lines with a unique labeling $s_0s_1s_2\dots$ for each line.

Let new symbols be defined from a combination of two symbols s_{t-1} and s_t , and such that the ordering of the Cantor set lines in the phase space is preserved. In table 1 new symbols $w_t \in \{0, 1\}$ are written together with the old symbols for the first 4 bounces. From the symbols $w_1 w_2 w_3 \dots$ a rational binary number $\gamma = 0.w_1 w_2 w_3 \dots = \sum_{t=1}^{\infty} w_t / 2^t$ is constructed. This symbolic coordinate γ is increasing with increasing values of x and increasing values of ϕ . To construct the symbols w_t let first v_t denote if two consecutive bounces s_{t-1} and s_t take place in a clockwise or anticlockwise direction. The symbols v_t are not well ordered because a bounce in a dispersing (concave) wall reverses the ordering in the phase space. At each bounce one has to invert the symbols v_t and this gives the algorithm:

$$\begin{aligned} v_t &= s_t - s_{t-1} \\ \text{if } v_t < 1 & \quad \text{then } v_t = v_t + 3 \\ w_t &= \begin{cases} v_t - 1 & \text{if } t \text{ odd} \\ 2 - v_t & \text{if } t \text{ even} \end{cases} \end{aligned} \tag{1}$$

The new symbols are constructed to reflect the ordering of the lines Λ_{∞}^+ in the phase space. Exchanging *odd* and *even* in algorithm (1) gives symbols $w'_t = 1 - w_t$, and a binary number $\gamma' = 0.w'_1 w'_2 w'_3 \dots$ decreasing with x and θ and therefore also the symbols w'_t are well ordered. From a orbit described by symbols s_t we have two mappings to the two well ordered symbols w_t and w'_t .

The particle bouncing at (x, ϕ) also has a past, and a symbol sequence describing this past. Let Λ_t^- be the union of points in phase space corresponding to a particle that has at least t bounces in the billiard before arriving at (x, ϕ) . Then Λ_1^- is the union of all points arriving at disk 1 after being bounced off disk 2 or off disk 3, and Λ_1^- is two diagonal strips in the phase space. The incoming angle has opposite sign to the outgoing angle and the strips in Λ_1^- have increasing ϕ with increasing x . We get the new symbols w_t from algorithm (1) with $t \leq 0$. The symbolic coordinate for the past is $\delta = 0.w_0 w_{-1} w_{-2} \dots = \sum_{t=1}^{\infty} w_{-t} / 2^t$. The coordinate δ is increasing with increasing value of x but δ is decreasing with increasing value of ϕ . If we choose the value γ' for the future we have $\delta' = 0.w'_0 w'_{-1} w'_{-2} \dots$ for the past.

Let $\Lambda_t = \Lambda_t^+ \cap \Lambda_t^-$. Then Λ_2 consist of $16 = (2 \cdot 2)^2$ areas in the phase space. Each of the 16 disjunct sets of Λ_2 has a unique enumeration by the 4 binary symbols $\{w_{-1}, w_0, w_1, w_2\}$. The set $\Lambda = \Lambda_{\infty}$ is a Cantor set and each point in the set is represented by a bi-infinite symbol string $\dots w_{-2} w_{-1} w_0 w_1 w_2 w_3 \dots$. The set Λ is the non-wandering set of the billiard.

The symbolic plane is the unit square $(\gamma, \delta); 0 \leq \gamma, \delta \leq 1$. Each non-wandering orbit is represented by a point in this plane, and each point in the plane is one of the non-wandering orbits. As mentioned earlier, we actually get two points in the symbol plane from one orbit. Because of the symmetries of rotation and reflection in the billiard, we get a number of different orbits in the billiard from one point in the symbol plane, but all the orbits have the same length and stability. The symbolic plane is a representation of the phase space of the well separated billiard,

with all gaps in the Cantor set removed. This is a very convenient space to work in; the curved lines in the phase space become straight lines and this space does not change if the distance r changes or if the borders of the disks are slightly changed as long as the symmetry and concavity is kept.

The billiard can be reduced to a fundamental domain^[19]. The fundamental domain is one sixth of the original 3 disk billiard and it is tiling the whole billiard. In our phase space the fundamental domain is the part with $x \leq \pi/6$ and is constructed in the symbol plane as follows: The symbol sequence $\dots s_{-2}s_{-1}s_0s_1s_2\dots$ gives two symbolic coordinates (γ, δ) and (γ', δ') but if $\gamma + \delta \neq 1$ then one point is in the fundamental region and the other point is in the region $x > \pi/6$. We find the two points (γ, δ) and (γ', δ') and choose (γ, δ) if $\gamma + \delta < 1$ or choose (γ', δ') if $\gamma + \delta > 1$. If $\gamma + \delta = 1$ then both (γ, δ) and (γ', δ') are on the border of the fundamental domain, and as our convention we choose (γ, δ) if $\gamma > \frac{1}{2}$ and choose (γ', δ') if $\gamma < \frac{1}{2}$. If we have a billiard without symmetry we only use one map from s_t to w_t and we do not have any fundamental domain.

2.2 Symbolic dynamics for N disks on a circle

As a generalization of the 3 disk billiard let N equal disks have the center of each disk on a large circle and let the distance between centers of neighbor disks on the large circle be r . Then Λ_1^+ is $(N - 1)$ strips in the phase space. The well ordered symbols $w_t \in \{0, 1, 2, \dots, (N - 2)\}$ are constructed from the anticlockwise enumeration of the disks $s_t \in \{1, 2, \dots, N\}$ using the algorithm

$$\begin{aligned} v &= s_t - s_{t-1} \\ \text{if } v_t < 1 & \quad \text{then } v_t = v_t + N \\ w_t &= \begin{cases} v_t - 1 & \text{if } t \text{ odd} \\ N - v_t - 1 & \text{if } t \text{ even} \end{cases} \end{aligned} \tag{2}$$

and the opposite ordered symbols are $w'_t = N - w_t - 2$. When $N = 3$ this is the same algorithm as (1), and for $N = 4$ this is the algorithm in ref.^[24]. From w_t we construct base $(N - 1)$ symbolic coordinates $\gamma = 0.w_1w_2w_3\dots = \sum_{t=1}^{\infty} w_t/(N - 1)^t$ and $\delta = 0.w_0w_{-1}w_{-2}\dots = \sum_{t=1}^{\infty} w_{1-t}/(N - 1)^t$ where $0 \leq \gamma, \delta \leq 1$, and in a analog way γ' and δ' .

The reduction to the fundamental domain in the symbol plane is the same as for 3 disk. If N is even there is one period 2 orbit with only one representation in the symbol plane $(\gamma, \delta) = (\gamma', \delta') = (1/2, 1/2)$.

2.3 N disks with a center disk

Let the billiard be a configuration of N disks on a large circle as in the billiard above and in addition one disk in the center of this large circle. The radius of the

disks are 1 and the distance between neighbor disks are r . The disks on the circle is enumerated anticlockwise from 1 to N and a bounce off the disk in the center is given the symbol $(N + 1)$. Figure 2 shows this configuration with $N = 6$. If the number of disks on the large circle is even, then, because of the disk in the center, a point particle can not bounce between two disks opposite to each other on the large circle. From the $(N + 1)$ symbols of the disks, we get $(N - 1)$ well ordered symbols, and Λ_1^+ consists of $(N - 1)$ strips in the phase space. With N (even) disks on the large circle and disk number $(N + 1)$ in the center of the large circle the algorithm defining the well ordered symbols $w_t \in \{0, 1, 2 \dots, (N - 2)\}$ is

$$\begin{aligned}
& \text{if } s_t = (N + 1) && \text{then } w_t = (N - 2)/2 \\
& \text{else if } s_{t-1} \neq (N + 1) && \text{then} \\
& \quad v_t = s_t - s_{t-1} \\
& \quad \text{if } v_t < 1 && \text{then } v_t = v_t + N \\
& \quad w_t = && \begin{cases} v_t - 1 & \text{if } t \text{ odd} \\ N - v_t - 1 & \text{if } t \text{ even} \end{cases} \\
& \text{else if } s_{t-1} = (N + 1) && \text{then} \\
& \quad v_t = s_t - s_{t-2} \\
& \quad \text{if } v_t < -(N - 2)/2 && \text{then } v_t = v_t + N \\
& \quad \text{if } v_t > (N - 2)/2 && \text{then } v_t = v_t - N \\
& \quad w_t = && \begin{cases} (N - 2)/2 + v_t & \text{if } t \text{ odd} \\ (N - 2)/2 - v_t & \text{if } t \text{ even} \end{cases}
\end{aligned} \tag{3}$$

The configuration with $N = 6$ can be looked at as a first step toward a description of the Lorentz gas^[25, 26], a triangular lattice with a hard disk in each lattice point and a point particle scattering in the lattice.

If the number of disks N on the large circle is odd, a point particle can reach all other disks after bouncing off one disk when the disks are sufficiently separated. The algorithm giving the symbols $w_t \in \{0, 1, 2 \dots, (N - 1)\}$ is

$$\begin{aligned}
& \text{if } s_t = (N + 1) && \text{then } w_t = (N - 1)/2 \\
& \text{else if } s_{t-1} \neq (N + 1) && \text{then} \\
& \quad v_t = s_t - s_{t-1} \\
& \quad \text{if } v_t < 1 && \text{then } v_t = v_t + N \\
& \quad w_t = && \begin{cases} v_t - 1 & \text{if } t \text{ odd} \\ N - v_t - 1 & \text{if } t \text{ even} \end{cases} \\
& \quad \text{if } w_t > (N - 1)/2 && \text{then } w_t = w_t + 1 \\
& \text{else if } s_{t-1} = (N + 1) && \text{then} \\
& \quad v_t = s_t - s_{t-2} \\
& \quad \text{if } v_t < -(N - 1)/2 && \text{then } v_t = v_t + N \\
& \quad \text{if } v_t > (N - 1)/2 && \text{then } v_t = v_t - N \\
& \quad w_t = && \begin{cases} (N - 1)/2 + v_t & \text{if } t \text{ odd} \\ (N - 1)/2 - v_t & \text{if } t \text{ even} \end{cases}
\end{aligned} \tag{4}$$

2.4 Orientation exchange at a bounce

One can explain why all the algorithms reverse the ordering of symbols at each bounce by a detailed description of a bounce. Assume two particles move along two arbitrarily close parallel lines and bounce off the dispersing (concave) wall at an angle $\phi \neq 0$ as in fig. 3. The particle that first hits the wall bounces off the wall along a new direction. The second particle crosses the first outgoing line and then bounces off the wall. The two outgoing lines cannot cross each other as long as the wall is concave or straight. The particle which is on the left side before the bounce, moves on the right side after the bounce, and this causes a change in the orientation at the bounce. If the angle $\phi = 0$, the orbits do not cross but because the velocity is reversed, the orbit that was on the left side before the bounce is on the right side after the bounce.

For a bounce off a focusing (convex) wall as in fig. 4, the two parallel orbits cross twice if $\phi \neq 0$. There is one crossing after the first particle has bounced off the wall and one crossing after the second particle has bounced off the wall. The particle which is on the left side before the bounce is on the left side also after the bounce, and the orientation does not change. This is true also for $\phi = 0$ when the orbits cross once after the bounces. These bounces take place e.g. for a point particle moving in the stadium billiard. Orbits bouncing only off the semicircles parts of the stadium walls can therefore be described by an ordered rotation number^[27]. The description of all orbits in the stadium billiard in well ordered symbols are given in ref.^[28].

2.5 A condition for sufficiently separated disks

We want a condition on the geometrical construction of the non-wandering set to distinguish between sufficiently separated disks and the case of pruning. We find that the disks are sufficiently separated if condition 1 below is true.

The definition of Λ_1^+ in the general case is as follows: Λ_1^+ consists of M strips and each strip m^+ is the union of points (x, ϕ) from which a straight line starting at x with angle ϕ hits a particular disk. As a straight line may go through other disks the M strips are not necessarily disjoint. In the same way we define Λ_1^- as the M strips where each strip m^- is the union of points (x, ϕ) where a line from point x with angle $-\phi$ hits a particular disk. We call the intersection of a strip m^+ and a strip m^- a rectangle. The set $\Lambda_1 = \Lambda_1^+ \cap \Lambda_1^-$ then consists of M^2 rectangles not necessarily disjoint. The construction of Λ_T with $T > 1$ follows from demanding that the outgoing angle at a bounce is equal to the incoming angle, but allowing the straight lines to go through a disc. Then Λ_T consists of M^{2T} rectangles.

Condition 1 *There exists a number $0 < T < \infty$ such that Λ_T consists of M^2 disjoint areas where each area is inside one of the M^2 rectangles of Λ_1 .*

The iteration of the M^2 disjoint areas corresponds to one more bounce and gives that each of the M^2 disjoint areas contains M^2 new disjoint areas. The M^4 rectangles of Λ_2 then contains M^4 disjoint areas. By induction we find that $\Lambda_{T+T'}$ gives $M^{2T'}$ disjoint areas inside the $M^{2T'}$ rectangles of $\Lambda_{T'}$. From this it follows that even if the rectangles of $\Lambda_{T'}$ overlap each other, the part of the non-wandering set belonging to the rectangles do not overlap. A symbol string $(w_{-T'+1}w_{-T'+2} \dots w_{T'-1}w'_T)$, with $w_t \in \{0, 1, \dots, (M-1)\}$ uniquely correspond to one rectangle when describing the part of the non-wandering set in this rectangle. It then follows that the disks are sufficiently separated.

If there does not exist a T according to the condition it may be that an infinitesimal change of the parameter gives a $0 < T < \infty$ and this is the critical parameter value where pruning starts. If non of the above is true, then the rectangles always overlap and there is pruning in the system and not all symbol strings corresponds to an orbit.

3 Pruning

When the distance r between the disks is small, a subset of the orbits are forbidden (not admissible). We conjecture that only two kinds of forbidden orbits exist in the dispersing billiards; orbits passing through a forbidden region (e.g. through a disk) and orbits going into a forbidden region and bouncing off the wall from the focusing side. Figure 5 a) shows a part of two *legal* orbits; one passing the dispersing wall and one bouncing off the wall on the dispersing side. Figure 5 c) shows a part of two *forbidden* orbits; one passing through the dispersing wall and the other going into the forbidden region and bouncing off the wall from the focusing side. Figure 5 b) shows that the limit orbit of both orbits is a line tangential to the disk. An orbit is therefore pruned together with at least one other orbit.

The orbits bouncing off the wall from the focusing side are orbits with $|\phi| > \pi/2$ in the phase space. In the phase space for sufficiently separated disks all parts of Λ have angle $|\phi| < \pi/2$. When r decreases, some points in Λ move closer to the lines $|\phi| = \pi/2$ and the pruning starts at r_c when the points in Λ with largest value of $|\phi|$ reach these lines.

When $r \rightarrow r_c$ with $r > r_c$, the value of T in condition 1 tends to ∞ . For $r < r_c$ the different rectangles of Λ_t always have some overlapping. This overlapping corresponds to forbidden regions in the symbol plane. The forbidden regions that are the simplest to describe are the two regions consisting of points in Λ with $|\phi| > \pi/2$. The borders of these regions in phase space are the lines $|\phi| = \pi/2$. In symbol plane the two regions correspond to two areas, one in the upper left corner and the other in the bottom right corner. The border is the symbolic representation of all points in Λ with $\phi = \pi/2$. To obtain this border, we scan through x values with the angle $\phi = \pm\pi/2$ and if $r > 2$ (an open system) in numerical work we only

keep points bouncing more than 20 bounces both in the future and in the past.

As both γ and δ increase with the value of x , the border is monotonously increasing in the symbol plane.

The forbidden regions described above contain only the forbidden orbits bouncing from the focusing side and not the orbits passing through a forbidden region. We choose to take the pre-image of this described region as our primary forbidden region and the pre-image of the border as our pruning front. The other family of forbidden orbits, that is orbits passing through a disk, is an other primary forbidden region and has a second pruning front that is very similar to the first pruning front. An orbit on this second pruning front has the same symbol sequence as an orbit on the first pruning front but without the one symbol for the tangential bounce. The two primary forbidden regions have a straight line as a common border and this line corresponds to a gap between two bands in Λ_1^+ for sufficiently separated disks. We consider the two primary forbidden regions to be bounded by one pruning front. This region can be described as the primary overlapping in Λ . In one of the examples we will show that there may be more than one of these primary forbidden regions, but all are constructed in the same way.

The iteration of the points in the symbol plane is a shift operation of the symbols. This shift operation is slightly more complicated for well ordered symbols than for the original disk symbols, see ref.^[22]. The primary forbidden region can by the shift operation be mapped forward and backward in time. The union of all images and preimages of the primary forbidden region is dense and takes the full measure in the symbol plane; Their complement, the union of the legal orbits has measure zero and is a Cantor set in the symbol plane.

From the primary forbidden region we can read off all the finite sequences that can make an orbit forbidden. If the well ordered alphabet has M symbols, we divide the symbol plane into rectangles with side length M^{-k} . Each rectangle has a unique labeling $w_{-k+1}w_{-k+2}\dots w_{k-1}w_k$ and corresponds to one disjoint area in the phase space of a sufficiently separated disk system. If the rectangle is inside the forbidden region, the symbol string $w_{-k+1}w_{-k+2}\dots w_{k-1}w_k$ is forbidden in the alphabet. By excluding all the forbidden symbol strings for a given number k we can redefine the symbolic dynamics in terms of a new alphabet with a new grammar, or in terms of a finite Markov graph^[8, 9]. This is an approximation of order k to the correct complete alphabet describing the billiard, and this gives an approximation to a Markov partition of the billiard. The implementation of such approximations in practice will be treated elsewhere.

3.1 Pruning for 3 disk billiard

The pruning in the 3 disk billiard starts when the points in Λ with the largest value of $|\phi|$ reach the value $|\phi| = \pi/2$. These outermost points in Λ are the hetroclinic

points created by the crossing of the unstable (stable) manifold of the fixed point $W = \bar{0}$ and the stable (unstable) manifold of the fixed point $W = \bar{1}$. The numerical value of the critical distance is $r_c = 2.04821419\dots$. The pruning front for 3 disks with $r < r_c$ is numerically determined by choosing points on disk 2 and starting the orbit by glancing off the disk at point x with the angle $\phi = \pm\pi/2$. The first bounce in one direction is off disk 1 and in the other direction the bounce is off disk 3. Let $s_0 = 1$, and a point on the first pruning front is determined by choosing $s_1 = 2$ and $s_2 = 3$. The other values of s_t are obtained from the numerical bouncing with starting point x on disk 2. To find a point on the second pruning front we do not include the bounce in disk 2 but assume the orbit grazes this disk. This gives the symbols $\tilde{s}_t = s_t$ for $t \leq 0$ and $\tilde{s}_t = s_{t+1}$ for $t > 0$. The points (γ, δ) are computed from the symbol sequences of orbits with starting point x between $2\pi/3$ and π . Plotting the points (γ, δ) in the symbol plane gives fig. 7 for parameter $r = 2$ (touching disks). The pruning front is a monotonously increasing line from $(0.375, 0)$ to $(0.5, 0.5)$ and a monotonously decreasing line from $(0.5, 0.5)$ to $(0.75, 0)$.

A numerical test of this pruning front can be performed by starting one orbit at a random point in the billiard, letting it bounce in the billiard for some time and calculate the point (γ, δ) for each bounce. In fig. 6 a) each bounce for an orbit with 10^6 bounces is marked by a point in the symbol plane. As expected, the primary forbidden region is white in fig. 6 a) as there are no bounces with these (γ, δ) values. In addition, there are 3 copies of this white region; the points (δ, γ) that are a result of the time reversion symmetry ($t \rightarrow -t$) and the points $(1-\gamma, 1-\delta)$ and $(1-\delta, 1-\gamma)$ that are a consequence of the symmetry between odd and even in algorithm (1). This last symmetry is the results of that our symbol plane includes two fundamental domains. All other white areas in fig. 6 a) are images or preimages of one primary forbidden region, where the iteration is a shift operation on the symbol string.

Fig. 7 and fig. 6 b) are magnifications of the symbol plane showing respectively the pruning front and the distribution of points of a long orbit. The long chaotic orbit visits points close to both pruning fronts but the probability is different as visualized in fig. 6 b). The orbit is often close to grazing a disk but very seldom does the orbit bounce with angle close to $\pi/2$. The tangential bounce is very unstable and this gives low probability density in the symbol plane to these bounces. The stability is however not sensitive to how close an orbit is to graze a wall, and the probability density in the symbol plane approaches the second pruning front rather uniformly.

One can convert the topological pruning front in fig. 7 into a symbolic description in different ways. The following method overcounts the number of allowed orbits.

In fig. 7 rectangles with side length 2^{-7} are plotted together with the pruning front. A rectangle with side length 2^{-k} in the symbol plane corresponds to a symbol string of length $2k$ in the symbolic description. Each rectangle in fig. 7 is identified with a symbol string $w_{-6}w_{-5}\dots w_6w_7$. If a rectangle is completely inside the primary forbidden region then the corresponding symbol string is forbidden. The

length 14 symbol strings $w_{-6}w_{-5}\dots w_6w_7$ that are forbidden for 3 touching disks are listed in column 2 in table 2. Finding the completely forbidden strings of length $2k$ for all $k \in \{1, 2, \dots\}$ gives a complete list of forbidden orbits. Table 2 gives this list for $k \leq 9$ when $r = 2$. The shortest forbidden symbol string is of length 12 ($k = 6$). In addition to the symbol strings w in table 2 the w' strings (obtained by letting $0 \rightarrow 1$ and $1 \rightarrow 0$) and the time reversed strings of w and w' have to be included as forbidden orbits.

It is possible to include in the table of forbidden strings the finite symbol strings describing rectangles that are just partly inside the primary forbidden region. This gives an undercounting of the legal orbits. It is also possible to apply a combination of these two methods. Other strategies are using some periodic orbits close to the pruning front as an approximation, or to use some number of points that are exactly on the pruning front. It is presently not clear which approach yields the most convergent sequence of approximate grammars.

The pruning front is not a continuous line in the symbol plane. The images and preimages of the pruning front cross the primary forbidden region and create open intervals in the pruning front. The result is that the pruning front itself is a Cantor set. The open intervals do not lead to any problems in converting the pruning front into a table of forbidden orbits. Any monotone curve in the interval, e.g. a straight line, gives a correct result.

The pruning front for an open 3 disk system with $r = 2.02$ is plotted in figure 8 and for a billiard with overlapping disks with $r = 1.97$ in figure 9. As expected the figures show that the primary pruned region increases when the distance r decreases. The structure of the pruning front does not change dramatically when the distance changes. In the billiard with overlapping disks, $r < 2$, the point particle can not bounce infinitely many times in a corner of the billiard and this gives an upper and lower bound on γ and δ : $0 < \beta \leq \gamma, \delta \leq (1 - \beta) < 1$. These strips of width β and their images and preimages are also forbidden regions. Figure 10 shows a long orbit with $r = 1.9$. A large area in the lower left corner and in the upper right corner are pruned by these strips.

3.2 Pruning for 4 disk billiard

The critical parameter value where the pruning start for the 4 disk system is $r_c = 2.20469453\dots$. The pruning front is similar to the pruning front for the 3 disk billiard. This billiard and the hyperbola billiard are discussed in detail in ref.^[24].

3.3 Pruning for a 6 + 1 disk system

The billiard with 6 disks on a large circle and one disk in the center has pruned orbits when the distance between the disks is less than $r_c = 3.59148407\dots$. Algorithm (3) with $N = 6$ gives the well ordered symbols for these system. Figure 2 shows the 7

disks and a part of one orbit that is grazing disk 7 (the center disk) and the symbolic coordinate (γ, δ) of this orbit is on the pruning front created by this center disk. The symbol plane used corresponds to the phase space of disk 1. In this system the disk 7 has a different phase space and symbol plane and the pruning front obtained is only valid for one of the disks on the large circle. There are two pruning fronts in the symbol plane; an orbit bouncing off disk 1 can graze either disk 7 as in figure 2 or it can graze disk 2. The primary pruned region in the symbol plane is limited by these pruning fronts as shown in figure 11 for parameter $r = 2.2$. One orbit grazes both disk 2 and disk 7 and this orbit gives a point connecting the two pruning fronts.

The primary pruned region is converted to a list of forbidden symbol strings similarly to the 3 disk billiard. Rectangles of size 5^{-k} correspond to symbol strings $w_{-k+1}w_{-k+2} \dots w_{k-1}w_k$ and the rectangles in the primary pruned region correspond to forbidden strings. Table 3 lists all the forbidden orbits of length 2 and 4.

4 Conclusions

We have shown that it is possible to define a well ordered covering alphabet for different concave billiard systems with a finite number of concave walls. For these systems the primary forbidden region is a simple connected region in a symbol plane. The border of this region is the pruning front, and we have found this pruning front numerically for the 3 disk system with different parameter values. The pruning front is also obtained for a more complicated billiard with 7 disks. The general method to construct a symbolic alphabet ordered the same way as the ordering of orbits in the phase space is discussed. A well ordered alphabet is necessary when finding the pruning front because, for any other alphabet the primary pruned region is a very complicated region.

We show how the topological information, the pruned region, can be converted into symbolic dynamics, a list of forbidden symbol strings. The tables list some of such forbidden strings.

The method can easily be applied to other finite concave billiards with similar symmetries as those considered in this article. Infinite billiards like the Bunimovich stadium billiard, the Sinai billiard and the Lorentz gas systems are more complicated because it is difficult to obtain a well ordered covering alphabet which is not pruned. These systems will be treated elsewhere. A smooth potential described by the same symbolic dynamics as a disk billiard system^[21, 29] may have a pruning front and will also be treated in a similar way. It is however problems in defining a bounce in the smooth potential and this is presently under investigation.

1. The three disks in configuration space enumerated anticlockwise.
2. The 6 + 1 disks, enumerated anticlockwise. $r = 2.2$. The orbit is tangent to disk 7, and therefore belongs to the pruning front for the system.
3. Show how two parallel orbits change orientation when bouncing in a concave wall.
4. Show that two parallel orbits do not change orientation bouncing in a convex wall.
5. The allowed, boundary and forbidden orbits: a) Two legal orbits. b) Two orbits on the pruning front. c) Two forbidden orbits.
6. The symbol plane picture of an orbit bouncing $5 \cdot 10^6$ times in the closed 3 disk system with touching disks (center distance $r = 2$). Each bounce corresponds to a point in the plane. a) the whole symbol plane. b) magnification of a part of the symbol plane showing the primary pruned region.
7. A magnification of the pruning front for 3 disks with $r = 2$, with line mesh spacing at 2^{-7} . The symbols give the symbolic past and the symbolic future of the rectangles.
8. The pruning front for 3 disks with $r = 2.02$.
9. The pruning front for 3 disks with $r = 1.97$
10. An orbit bouncing $2 \cdot 10^6$ times in the closed 3 disk system with center distance $r = 1.9$.
11. The pruning front for 6+1 disks, $r = 2.2$.

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| $s_0s_1s_2s_3s_4$ | $w_1w_2w_3w_4$ |
|-------------------|----------------|
| 12121 | 0000 |
| 12123 | 0001 |
| 12132 | 0010 |
| 12131 | 0011 |
| 12313 | 0100 |
| 12312 | 0101 |
| 12321 | 0110 |
| 12323 | 0111 |
| 13232 | 1000 |
| 13231 | 1001 |
| 13213 | 1010 |
| 13212 | 1011 |
| 13121 | 1100 |
| 13123 | 1101 |
| 13132 | 1110 |
| 13131 | 1111 |

Table 1: The table shows the ordering of 4 bounces of a particle starting on disc 1 at time 0. The symbol s_t is the number of the disc, while w_t is new symbols reflecting the ordering in phase space.

| $\neg w_{-5} \dots w_{6-}$ | $\neg w_{-6} \dots w_{7-}$ | $\neg w_{-7} \dots w_{8-}$ |
|----------------------------|----------------------------|----------------------------|
| 000000 · 100000 | 0000000 · 1000011 | 00000000 · 10001010 |
| | 0000000 · 1000010 | 00000000 · 10001001 |
| | 0100000 · 1000001 | 00000000 · 10001000 |
| | 0100000 · 1000000 | 01000000 · 10000101 |
| | 1100000 · 1000000 | 01000000 · 10000100 |
| | 0010000 · 1000000 | 11000000 · 10000100 |
| | 1000000 · 0111111 | 01100000 · 10000010 |
| | 0000000 · 0111111 | 11100000 · 10000010 |
| | 0000000 · 0111110 | 01010000 · 10000001 |
| | 0100000 · 0111111 | 01010000 · 10000000 |
| | | 11010000 · 10000000 |
| | | 00110000 · 10000000 |
| | | 10110000 · 10000000 |
| | | 01110000 · 10000000 |
| | | 11110000 · 10000000 |
| | | 00001000 · 10000000 |
| | | 01010000 · 01111111 |
| | | 10010000 · 01111111 |
| | | 00010000 · 01111111 |
| | | 11100000 · 01111111 |
| | | 01100000 · 01111111 |
| | | 01000000 · 01111101 |
| | | 00000000 · 01111011 |

Table 2: The forbidden orbits in 3 discs, $r = 2$. “Under counting” approximation to level 9.

| $_w_8 \dots w_9_$ | | |
|-----------------------|-----------------------|-----------------------|
| 000000000 · 011110011 | 110101000 · 100000000 | 001000000 · 100001110 |
| 000000000 · 011110100 | 010101000 · 100000000 | 001000000 · 100001111 |
| 000000000 · 011110101 | 100101000 · 100000000 | 110000000 · 100010000 |
| 100000000 · 011110101 | 000101000 · 100000000 | 010000000 · 100010000 |
| 010000000 · 011110111 | 111001000 · 100000000 | 010000000 · 100010001 |
| 001000000 · 011111000 | 011001000 · 100000000 | 000000000 · 100010110 |
| 001000000 · 011111001 | 101001000 · 100000000 | 000000000 · 100010111 |
| 011000000 · 011111011 | 001001000 · 100000000 | 000000000 · 100011000 |
| 111000000 · 011111011 | 110001000 · 100000000 | 000000000 · 100011001 |
| 000100000 · 011111011 | 010001000 · 100000000 | |
| 001100000 · 011111101 | 000110000 · 100000010 | |
| 101100000 · 011111101 | 111010000 · 100000010 | |
| 011100000 · 011111101 | 011010000 · 100000010 | |
| 111100000 · 011111101 | 011010000 · 100000011 | |
| 011010000 · 011111110 | 010010000 · 100000100 | |
| 011010000 · 011111111 | 100010000 · 100000100 | |
| 111010000 · 011111111 | 000010000 · 100000100 | |
| 000110000 · 011111111 | 000010000 · 100000101 | |
| 100110000 · 011111111 | 001100000 · 100000110 | |
| 010110000 · 011111111 | 010100000 · 100001000 | |
| 110110000 · 011111111 | 100100000 · 100001000 | |
| 001110000 · 011111111 | 000100000 · 100001000 | |
| 101110000 · 011111111 | 000100000 · 100001001 | |
| 011110000 · 011111111 | 111000000 · 100001010 | |
| 111110000 · 011111111 | 011000000 · 100001010 | |
| 000001000 · 011111111 | 001000000 · 100001100 | |
| 100001000 · 011111111 | 001000000 · 100001101 | |

Table 2: Continue.

| $w_0 w_1$ | w_{-1} | $\dots w_2$ | | | | |
|-------------|---------------|---------------|---------------|---------------|---------------|---------------|
| $4 \cdot 1$ | $04 \cdot 20$ | $43 \cdot 20$ | $32 \cdot 11$ | $12 \cdot 14$ | $11 \cdot 14$ | $00 \cdot 14$ |
| $3 \cdot 1$ | $14 \cdot 20$ | $42 \cdot 20$ | $32 \cdot 12$ | $02 \cdot 13$ | $21 \cdot 14$ | $10 \cdot 14$ |
| | $24 \cdot 20$ | $32 \cdot 20$ | $32 \cdot 13$ | $02 \cdot 14$ | $31 \cdot 14$ | $20 \cdot 14$ |
| | $34 \cdot 20$ | $42 \cdot 10$ | $32 \cdot 14$ | $01 \cdot 13$ | $41 \cdot 14$ | $30 \cdot 14$ |
| | $44 \cdot 20$ | $42 \cdot 11$ | $22 \cdot 11$ | $11 \cdot 13$ | $00 \cdot 13$ | $40 \cdot 14$ |
| | $03 \cdot 20$ | $42 \cdot 12$ | $22 \cdot 12$ | $21 \cdot 13$ | $10 \cdot 13$ | $10 \cdot 10$ |
| | $13 \cdot 20$ | $42 \cdot 13$ | $22 \cdot 13$ | $31 \cdot 13$ | $20 \cdot 13$ | $00 \cdot 10$ |
| | $23 \cdot 20$ | $42 \cdot 14$ | $22 \cdot 14$ | $41 \cdot 13$ | $30 \cdot 13$ | $00 \cdot 11$ |
| | $33 \cdot 20$ | $32 \cdot 10$ | $12 \cdot 13$ | $01 \cdot 14$ | $40 \cdot 13$ | |

Table 3: The forbidden orbits in 6+1 discs, $r = 2.2$. “Under counting” approximation to level 2.